5. Series developments

5.1 Disjunctive series development

THEOREM
Every Boolean function can be developed in a sum of products.

PROOF
By means of the first distributive property, all parentheses of the type $(a+b)(a+c) = a+bc$ can be removed [b and c may be complex functions].
By means of the second distributive property, all parentheses of the type $a.(b+c) = ab+ac$ can be removed.
Repeated application leads to a sum of products. This is called a **disjunctive** development.
Example: $abc + a\bar{c}d + \bar{b}d$

THEOREM
Any disjunctive form can be written as a sum of products in which <u>in each product all variables</u> occur.

PROOF
If in a form x the variable p is missing, then $x.1 = x.(p + \bar{p}) = xp + x\bar{p}$. So p has been added.
Example: suppose $\alpha = abc + acd + bd$
$abc = abc1 = abc(d + \bar{d}) = abcd + abc\bar{d}$
$acd = acd1 = acd(b + \bar{b}) = abcd + a\bar{b}cd$
$bd = bd1 = bd(ac + a\bar{c} + \bar{a}c + \bar{a}\bar{c}) = abcd + ab\bar{c}d + \bar{a}bcd + \bar{a}b\bar{c}d$
Hence, $\alpha = abcd + abc\bar{d} + ab\bar{c}d + a\bar{b}cd + \bar{a}bcd + \bar{a}b\bar{c}d$.
This form is called the **disjunctive normal form**.

If we write a function as a sum of products, this can be developed as follows:

$$T(a, b, c, \cdots) = (a + \bar{a})T = a\phi_1(b, c, \cdots) + \bar{a}\phi_2(b, c, \cdots)$$

ϕ_1 is a sum of products where a has been brought outside the parentheses.
ϕ_2 similar for $\bar{a}$.
Suppose $a = 1$, (i.e. $\bar{a} = 0$) and substitute this in $T(a, b, c, \cdots)$.

$$T(1, b, c, \cdots) = \phi_1(b, c, \cdots) \qquad \phi_1 \quad \text{is determined}$$

Now suppose $a = 0$, (i.e. $\bar{a} = 1$):

$$T(0, b, c, \cdots) = \phi_2(b, c, \cdots) \qquad \phi_2 \quad \text{is determined}$$

Hence:

$$T(a, b, c, \cdots) = aT(1, b, c, \cdots) + \bar{a}T(0, b, c, \cdots)$$

Repeated application gives

$$T(a, b, c, \cdots) = abT(1, 1, c, \cdots) + a\bar{b}T(1, 0, c, \cdots) + \bar{a}bT(0, 1, c, \cdots) + \bar{a}\bar{b}T(0, 0, c, \cdots)$$

If there are n variables a,b,c,...,n, the n-th variable can be assigned the weight $2^0 = 1$ and the variable a the weight 2^{n-1}. We can now write

$$T(a, b, \cdots, n) = \sum_{i=0}^{2^n-1} g_i m_i$$

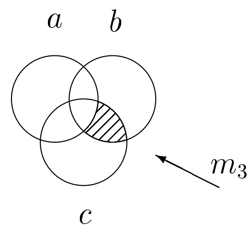
For three variables is $m_2 = \bar{a}b\bar{c}$: a has weight 4, b has weight 2 and c has weight 1. Inversion implies that a variable does not contribute to the sum. So $m_2 = \bar{a}b\bar{c} = 0.2^2 + 1.2^1 + 0.2^0 = 2$.

• The terms g_i denote a $T(1, 0, 1, \cdots)$. So is $g_2 = T(0, 1, 0)$, i.e. the value (0 or 1) of the function $T(a, b, c)$ for $a = 0$, $b = 1$, $c = 0$.

a	b	c	
0	0	0	g_0
0	0	1	g_1
0	1	0	g_2
0	1	1	g_3
1	0	0	g_4
1	0	1	g_5
1	1	0	g_6
1	1	1	g_7

• The terms m_i are called **min terms** and they represent in a Venn diagram <u>one</u> intersection:

$m_0 = \bar{a}\bar{b}\bar{c}$
$m_1 = \bar{a}\bar{b}c$
$m_2 = \bar{a}b\bar{c}$
$m_3 = \bar{a}bc$
$m_4 = a\bar{b}\bar{c}$
$m_5 = a\bar{b}c$
$m_6 = ab\bar{c}$
$m_7 = abc$



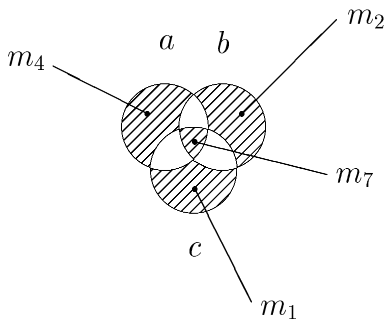
We see that $m_i.m_j = 0$ for $i \neq j$ and the min terms cover piece by piece the Venn diagram and represent a function.

• Relation between the function table and the series development:
The **inputs** of a function (truth) table correspond with the min terms.
The **outputs** are then the g terms of the series development.

Example:
$a \oplus (b \oplus c) =$
$a.\overline{(b \oplus c)} + \bar{a}.(b \oplus c) =$

$a(bc+\bar{b}\bar{c})+\bar{a}(b\bar{c}+\bar{b}c)=$
$abc+a\bar{b}\bar{c}+\bar{a}b\bar{c}+\bar{a}\bar{b}c=$
$m_7+m_4+m_2+m_1$.
This function is represented by the following Venn diagram and truth table:

a	b	c	
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1



5.2 Conjunctive series development

THEOREM
Every Boolean function can be developed in a product of sums.

PROOF
According to the distributive property, $a+bc=(a+b)(a+c)$. This can be done recursively. By adding e.g. $d\bar{d}$, the variable d can be introduced:
$(a+b)(a+c)+d\bar{d}=(d+(a+b)(a+c))(\bar{d}+(a+b)(a+c))=$
$(d+a+b)(d+a+c)(\bar{d}+a+b)(\bar{d}+a+c)$.

Hence, any function can be written as a product of sums where <u>in each sum all variables</u> occur. This is called the **conjunctive normal form**.

Again, a and $\bar{a}$ can be brought outside parentheses (distrib property):

$$T(a,b,c)=[a+\psi_1(b,c)][\bar{a}+\psi_2(b,c)]$$

Suppose
$a=0,\bar{a}=1\rightarrow T(0,b,c)=\psi_1(b,c)$ and
$a=1,\bar{a}=0\rightarrow T(1,b,c)=\psi_2(b,c)$
such that $T(a,b,c)=[a+T(0,b,c)][\bar{a}+T(1,b,c)]$.
Repeated application for b and c gives:
$T(a,b,c)=$
$[a+b+c+T(0,0,0)]\cdot[a+b+\bar{c}+T(0,0,1)]\cdot$
$[a+\bar{b}+c+T(0,1,0)]\cdot[a+\bar{b}+\bar{c}+T(0,1,1)]\cdot$
$[\bar{a}+b+c+T(1,0,0)]\cdot[\bar{a}+b+\bar{c}+T(1,0,1)]\cdot$
$[\bar{a}+\bar{b}+c+T(1,1,0)]\cdot[\bar{a}+\bar{b}+\bar{c}+T(1,1,1)]$

In the same way as for a disjunctive series development we get

$$T(a,b,\cdots,n)=\prod_{i=0}^{2^n-1}(G_i+M_i)$$

Alternative derivation:

$$T(a,b,\cdots,n)=\sum_{i=0}^{2^n-1} g_i m_i$$

$\bar{T}$ implies all $0\to 1$ and all $1\to 0$. So

$$\bar{T}(a,b,\cdots,n)=\sum_{i=0}^{2^n-1} \bar{g}_i m_i$$

According to de Morgan, $\overline{\sum\phi}=\prod\overline{\phi}$ and $\overline{\prod\phi}=\sum\overline{\phi}$. Hence

$$(\overline{T})\prime = T = \overline{\sum_{i=0}^{2^n-1} \bar{g}_i m_i} = \prod_{i=0}^{2^n-1}(g_i+\bar{m}_i).$$

$$\rightsquigarrow \quad \bar{m}_i = M_j \;\text{ with }\; j=2^n-1-i.$$

- A term with $g_i=1$ becomes 1 in the product (invisible).
- A term with $g_i=0$ becomes M_j in the product (it exists!)
- The terms M_j are called **max terms** and they are the inverse of the min terms.

In the literature there exist two different notations:
1st: $a+b+c=M_7$
2nd: $a+b+c=M_0$
The 2nd takes over the notation for g_i, $[a+b+c+T(0,0,0)]$.
Here we use the 1st notation $a+b+c=M_7$, such that

$$T(a,b,c)=\prod_{i=0}^{2^n-1}(G_i+M_{2^n-1-i}).$$

Notation: instead of the complete min/max terms, only the subscripts are used

$$\sum m_1,m_3,m_7 \equiv \sum 1,3,7$$

$$\prod M_1,M_3,M_7 \equiv \prod 1,3,7$$

Conversion:
To find the max terms from the min terms, we must determine the min terms for which $g_i=0$ because for these the $M_j=M_{2^n-1-i}$ will exist.

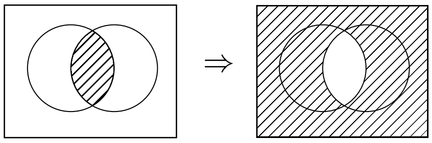
Example:
We have 3 variables and $T(a,b,c)=\sum(1,3,7)$.
The missing min terms are 0, 2, 4, 5, 6. The max terms are $M_{7-0},M_{7-2},M_{7-4},M_{7-5},M_{7-6}$, or M_7,M_5,M_3,M_2,M_1
$T(a,b,c)=\sum(1,3,7)=\prod(1,2,3,5,7)$.

Verify: $\sum(0,1,3,5)=\prod(0,1,3,5)$

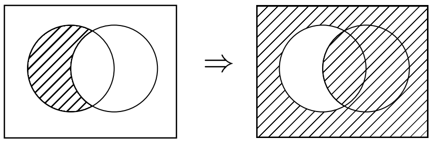
Summarising:
- The smallest sets (smallest areas in a Venn diagram) are min terms (min = minimal).
- The negation of the smallest gives the largest: max terms.

minterm maxterm

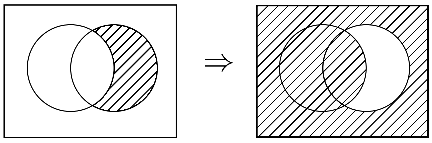
$(a.b)' = \bar{a}.b + a.\bar{b} + \bar{a}.\bar{b} = \bar{a} + \bar{b}$



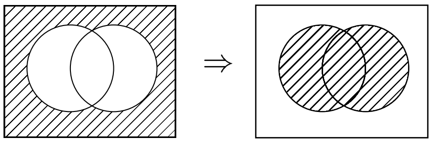
$(a.\bar{b})' = a.b + \bar{a}.b + \bar{a}.\bar{b} = \bar{a} + b$



$(\bar{a}.b)' = a.b + a.\bar{b} + \bar{a}.\bar{b} = a + \bar{b}$



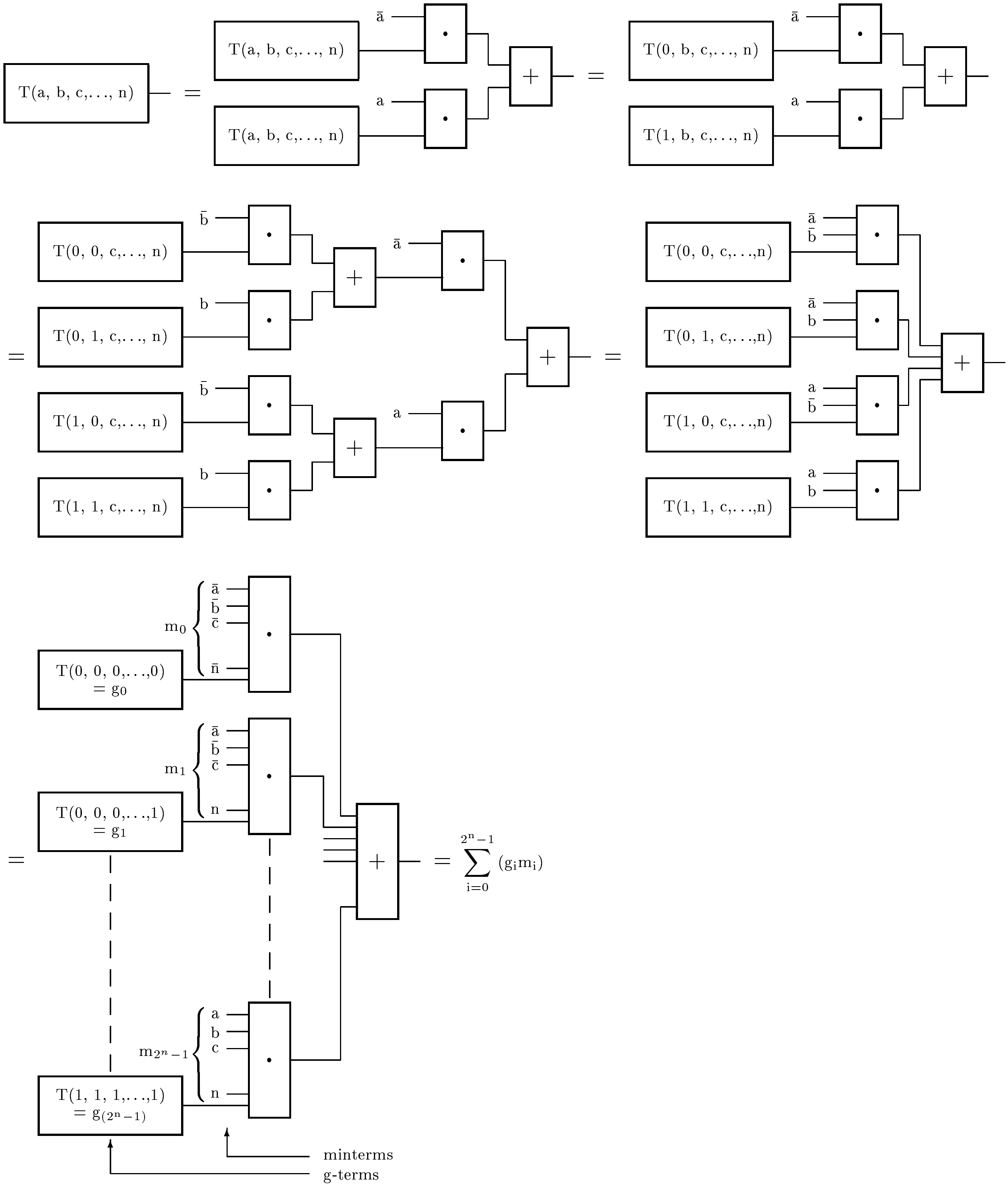
$(\bar{a}.\bar{b})' = a.b + \bar{a}.b + a.\bar{b} = a + b$



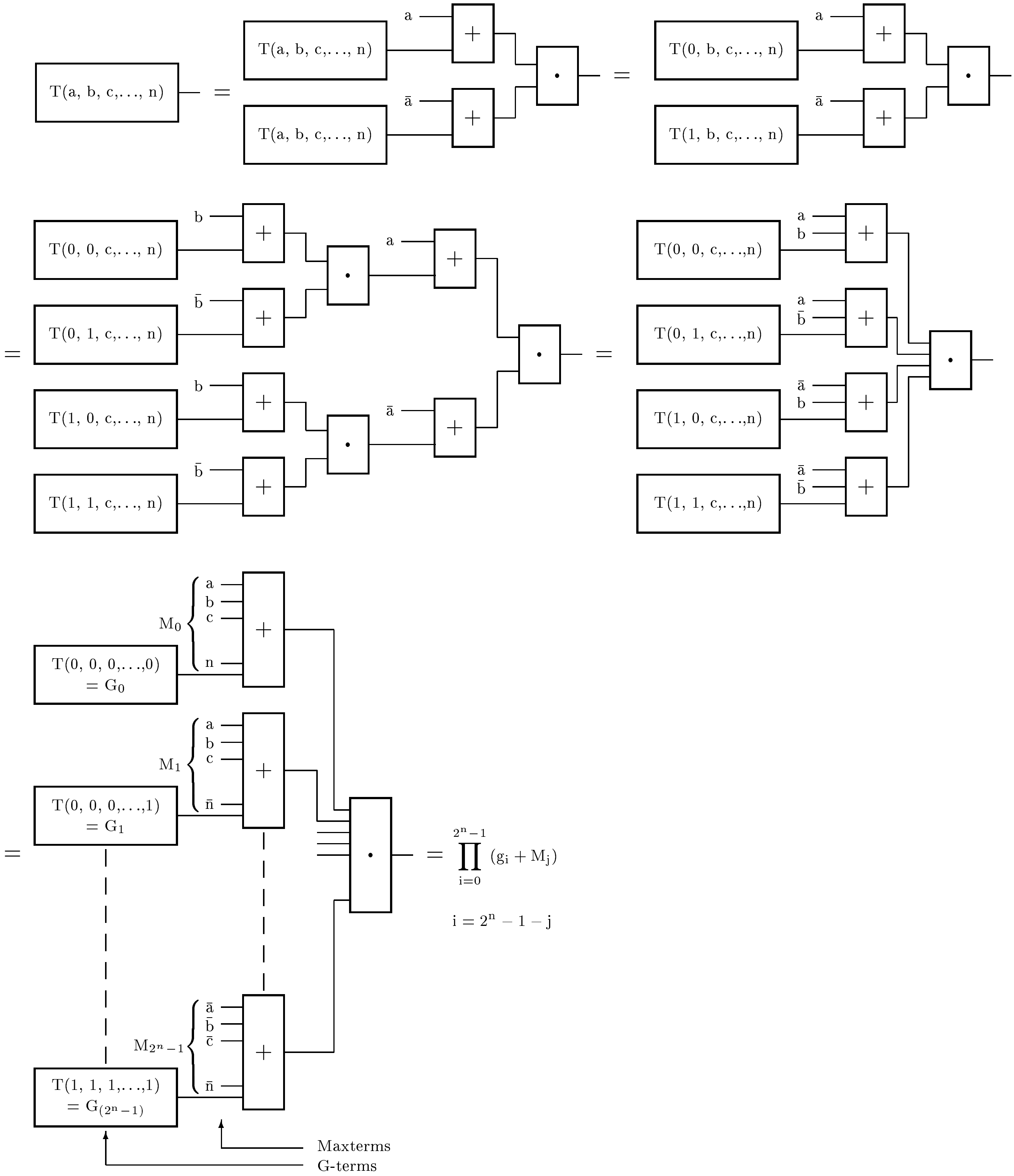
• Disjunctive series are mainly used to simplify circuits with AND, OR and/or NAND gates.
• Conjunctive series are mainly used to simplify circuits with NOR gates.

Comparison/Summary:

Disjunctive	Conjunctive
$T(a, b, \cdots, n) = \sum_{i=0}^{2^n-1} g_i m_i$	$T(a, b, \cdots, n) = \prod_{i=0}^{2^n-1}(G_i + M_j)$ $j = 2^n - 1 - i$
A 0 in one of the g terms corresponds with a negation in the coupled m term: $g_3 = T(0, 1, 1) \Leftrightarrow m_3 = \bar{a}bc$	A 1 in one of the G terms corresponds with a negation in the coupled M term: $G_3 = T(0, 1, 1) \Leftrightarrow M_4 = a + \bar{b} + \bar{c}$
If the number of terms $g_i m_i$ with $g_i = 1$ equals k, the function can be written as the sum of k products.	If the number of terms $g_i m_i$ with $g_i = 1$ equals k, the function can be written as the product of $2^n - k$ factors.
For each combination of the variables, <u>at most</u> one of the k terms of the sum is equal to 1: $\Rightarrow m_p m_q = 0$ $(p \neq q)$.	For each combination of the variables, <u>at most</u> one of the $2^n - k$ factors of the product is equal to 0: $\Rightarrow M_p + M_q = 1$ $(p \neq q)$.



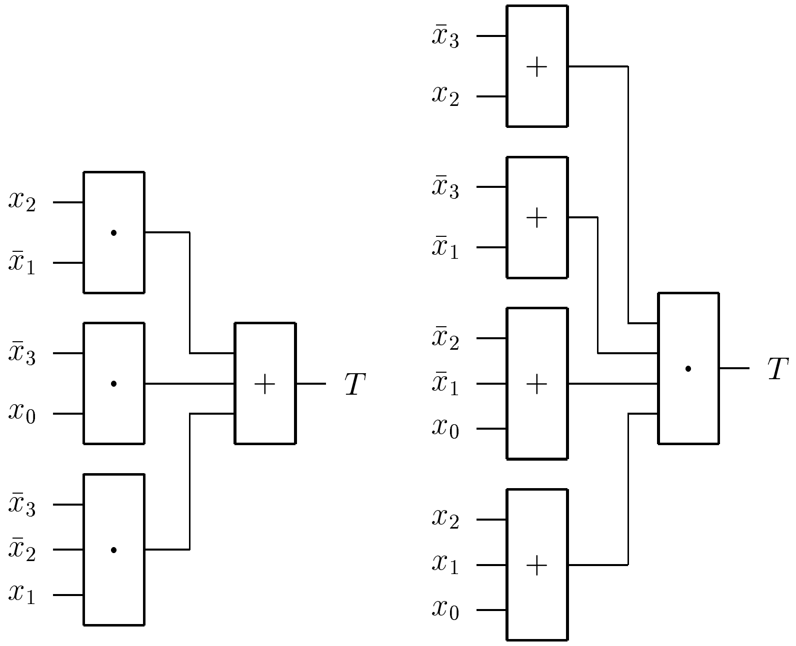
This scheme represents a disjunctive series development.



This scheme represents a conjunctive series development.

5.3 Conversion AND/OR to NAND/NOR circuits

Example to be used below:



This is the **regular form**: only ANDs and ORs:

$$T = \sum(1,2,3,4,5,7,12,13) = x_2\bar{x}_1 + \bar{x}_3 x_0 + \bar{x}_3\bar{x}_2 x_1$$

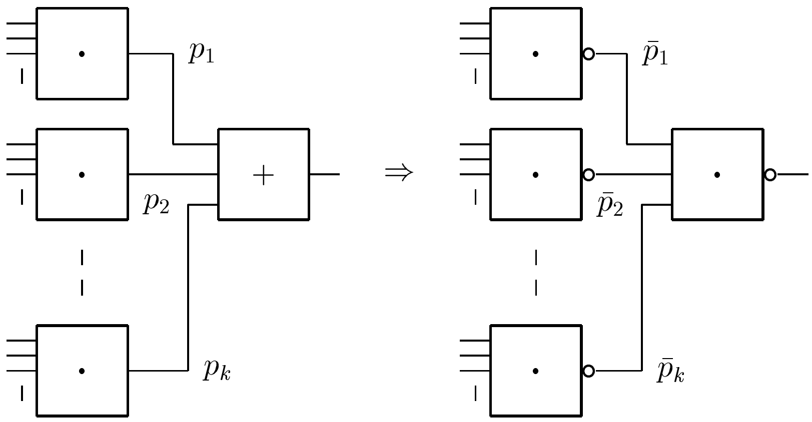
which equals

$$T = \prod(0,1,4,5,6,7,9,15) = (\bar{x}_3 + x_2)(\bar{x}_3 + \bar{x}_1)(\bar{x}_2 + \bar{x}_1 + x_0)(x_2 + x_1 + x_0).$$

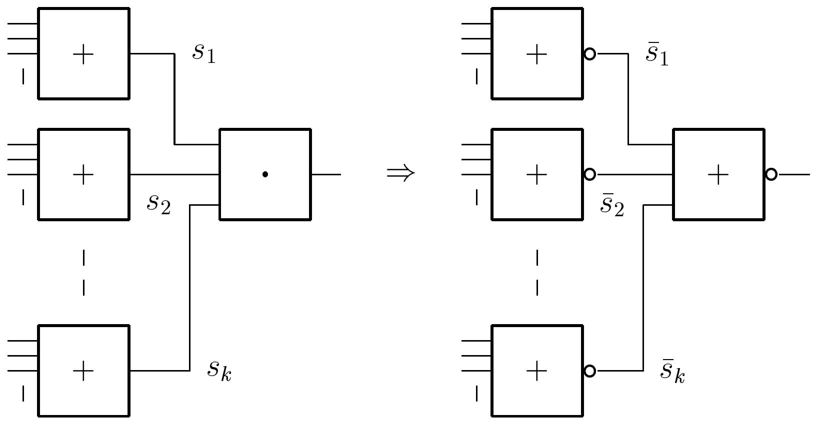
• A logical function T in disjunctive form can be realised with a 2–stage circuit consisting of NANDs only. Algebraically:

$$\sum g_i m_i = \overline{\prod \overline{g_i m_i}} = \overline{\prod (\bar{g}_i + \bar{m}_i)}$$

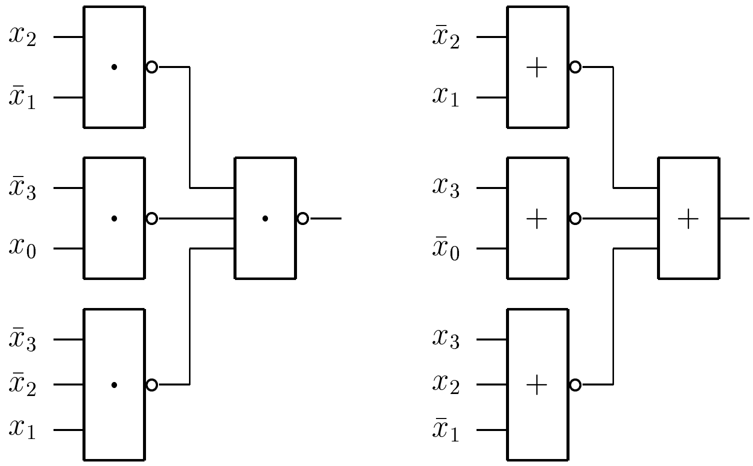
In the figure below $p_1 + p_2 + p_3 + \cdots p_k = \overline{\bar{p}_1\bar{p}_2\bar{p}_3\cdots\bar{p}_k}$.



• A logical function T in conjunctive form can be realised with a 2–stage circuit consisting of NORs only:



The above example (top previous page) can be realised with 4 NAND gates (see below left), or with 5 NOR gates (try yourself). If this function must be realised with NOR gates, a more simple circuit is obtained by making $\bar{T}$ in conjunctive form and by using an OR instead of the last NOR:



In the left circuit we have

$$T = x_2\bar{x}_1 + \bar{x}_3x_0 + \bar{x}_3\bar{x}_2x_1$$

such that

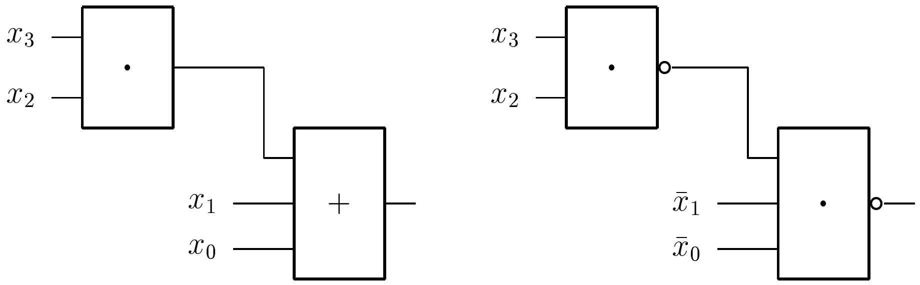
$$\bar{T} = (\bar{x}_2 + x_1)(x_3 + \bar{x}_0)(x_3 + x_2 + \bar{x}_1).$$

Taking the double negation of T, we obtain

$$T = \overline{(\bar{x}_2 + x_1)(x_3 + \bar{x}_0)(x_3 + x_2 + \bar{x}_1)},$$

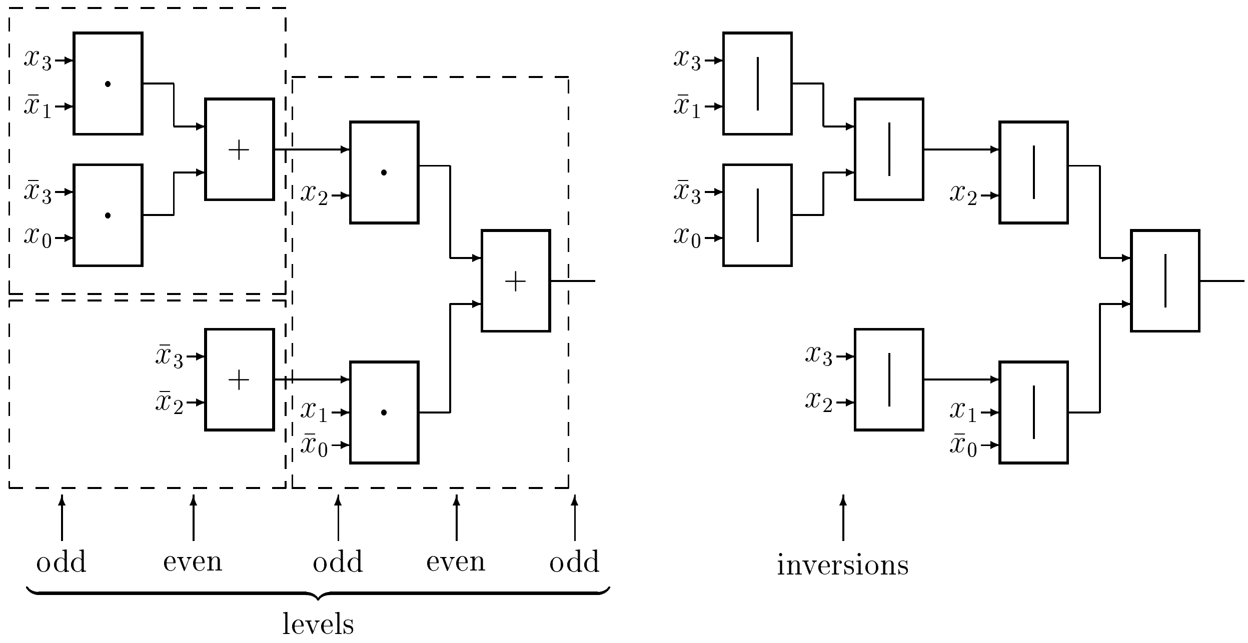
which corresponds to the right circuit.

• If one or more of the products in the disjunctive form consists of only one variable, then the corresponding AND gate is not necessary. Converting to NAND gates, this gate reduces to an invertor which can be omitted by taking the inverse of the variable (in practice: if the inverse variable is already available in a circuit):



$$T = x_3x_2 + x_1 + x_0$$

• This inversion must also be applied in converting the conjunctive form to NOR gates, that is, if a *sum* consists of only one variable. See next page.



The output of the first circuit is given by T $= x_2(x_3\bar{x}_1 + \bar{x}_3 x_0) + x_1\bar{x}_0(\bar{x}_3 + \bar{x}_2)$. Simplifying this expression, by increasing the "depth" of the circuit, we obtain an expression that allows to construct the circuit to the left. This circuit can be easily realized by using only NANDs, in which case the inputs of the even levels need to be inverted. It is also possible to realize the same function using only NORs and one OR. This is shown in the lower figure. Note the difference between the "numbering" of the levels (even and odd). *Check this trick!*