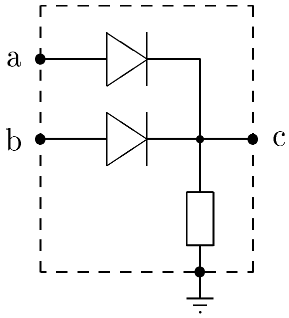
4. Physical elements and logical functions

Take a box (a *blackbox*) with three connectors a, b and c, and a fourth one for *ground.* On a and on b we can put a voltage (positive) and we measure the voltage on c. Call voltage=H and no voltage=L. H comes from high, L from low. Using simple electronic components like diodes[1], resistors and transistors, we will introduce a few basic logical functions.

In the scheme below, a current can only flow from left to right. If the voltage at input a is positive, a current flows through the upper diode and through the resistor. If, at the same time, input b is connected to ground (0 V), the current that goes through the upper diode cannot flow through the lower diode. Hence, if a=L and b=L no current flows through the resistor. As a consequence, we measure c=L. If a=H and b=L, a current flows through the resistor and c=H. So we can measure the output c for all possible combinations H and L on the inputs a and b. We then get the left table:

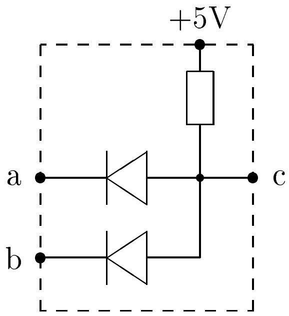


a	b	c
L	L	L
L	H	H
H	L	H
H	H	H

a	b	c
0	0	0
0	1	1
1	0	1
1	1	1

If we map {H,L} one-to-one on the set {1,0}, such that H $\rightarrow$ 1 and L $\rightarrow$ 0, we get the right table, which we call a **truth table**. From this we see that a+b=c. We call this the **OR function**.

Another "black-box," but now with an input connected to + or H instead of to ground:



a	b	c
L	L	L
L	H	L
H	L	L
H	H	H

a	b	c
0	0	0
0	1	0
1	0	0
1	1	1

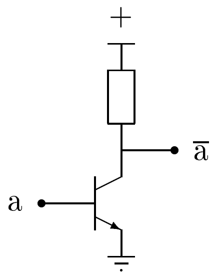
If both inputs a and b are H, there is no current through the resistor and c will be H. However, if one or both of the inputs a and b is L, there will be a current and the voltage on c drops to L (in this scheme, a current can only flow from right to left; take a look at the diode definition). From the right table we see that this circuit realises the

[1] A diode is a nonlinear electronic component that can conduct current in one direction but not in the other. The symbol's arrow shows the way that the current can go to.

function a.b=c. We call this the **AND function**. So we have seen that two postulates of the Boole algebra can be realised physically.

Another postulate is $a.\bar{a} = 0$ together with $a + \bar{a} = 1$. We can realise $\bar{a}$ with a transistor (which is used here as a switching element, i.e. a controllable diode):

If a is L, the transistor does not conduct and $\bar{a}$, the output, is H.
If a is H, the transistor conducts, there is a voltage drop over the resistor R and $\bar{a}$ is L.



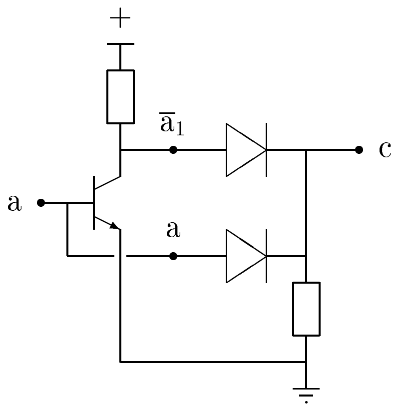
This gives the tables

a	$\bar{a}$
L	H
H	L

and

a	$\bar{a}$
0	1
1	0

If we connect a and $\bar{a}$ to a circuit that realises $a + b$, the OR circuit described above, we get:



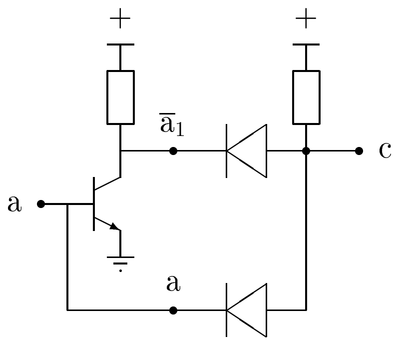
a	$\bar{a}_1$	c
L	H	H
H	L	H

LL and HH cannot occur!

a	$\bar{a}_1$	c
0	1	1
1	0	1

Hence $a + \bar{a}_1 = 1$

Similarly, the postulate $a\bar{a} = 0$ can be realised by taking the AND circuit:



a	$\bar{a}_1$	c
L	H	L
H	L	L

LL and HH cannot occur!

a	$\bar{a}_1$	c
0	1	0
1	0	0

Hence $a\bar{a}_1 = 0$

This proves that $a + \bar{a}_1 = 1$ and $a\bar{a}_1 = 0$. In other words, $\bar{a}_1$ exists and equals $\bar{a}$. In the same way the other postulates can be proved. For example, the associative property $(a + b) + c = a + (b + c)$ requires two OR circuits, normally called OR *gates,* in series. The distributive property $a + bc = (a + b)(a + c)$ requires one AND gate and one OR gate to realise the left term and two OR gates and one AND gate for the right term.

In all cases the truth tables for the left and right terms must be equal.

4.1 Function overview

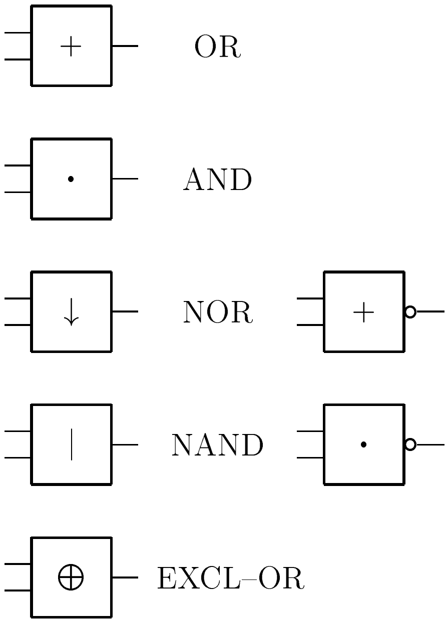
With two variables multiple functions can be realised, 16 in total. One of these is $f(a,b) = 0$, i.e. a circuit that is always open. Another one is $f(a,b) = 1$, i.e. a circuit that is always closed. Table 1 lists all functions of two variables. It consists of $2^2 = 4$ columns with the input combinations 00, 01, 10 and 11. In general, for n variables there are 2^n columns.
In the table we see that there are first zero 1's: C_0^4
then one 1 in 4 positions: C_1^4
then two 1's in 6 positions: C_2^4
then three 1's in 4 positions: C_3^4
then four 1's in 1 position: C_4^4
Together $C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4 = 2^4 = 16$ functions possible.

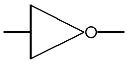
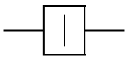
With n variables $\sum_{i=0}^{2^n} C_i^{2^n} = 2^{2^n}$.
For $n = 3 \rightarrow 256$
For $n = 4 \rightarrow 65{,}536$
For $n = 5 \rightarrow 4{,}294{,}967{,}296$
For $n = 6 \rightarrow \infty$

a	1	0	1	0		
b	1	1	0	0		
T_0	0	0	0	0	zero function (open, false)	$\bar{T}_{15}$
T_1	0	0	0	1	$\bar{a}\bar{b}$, $a \downarrow b$, $\overline{a+b}$, NOR, Peirce function	$\bar{T}_{14}$
T_2	0	0	1	0	$a.\bar{b}$, $a \not\rightarrow b$, inhibition, exclusion	$\bar{T}_{13}$
T_4	0	1	0	0	$\bar{a}.b$, $b \not\rightarrow a$, inhibition, exclusion	$\bar{T}_{11}$
T_8	1	0	0	0	$a.b$, AND, conjunction	$\bar{T}_7$
T_3	0	0	1	1	$\bar{b}$, $\neg b$, negation, inversion of b	$\bar{T}_{12}$
T_5	0	1	0	1	$\bar{a}$, $\neg a$, negation, inversion of a	$\bar{T}_{10}$
T_6	0	1	1	0	$a\bar{b} + \bar{a}b$, $a \oplus b$, $a \not\equiv b$, EXCLUSIVE–OR	$\bar{T}_9$
T_9	1	0	0	1	$ab + \bar{a}\bar{b}$, $a \odot b$, $a \equiv b$, equal	$\bar{T}_6$
T_{10}	1	0	1	0	a identity	$\bar{T}_5$
T_{12}	1	1	0	0	b identity	$\bar{T}_3$
T_7	0	1	1	1	$\bar{a} + \bar{b} = \overline{a.b}$, $a\|b$, NAND, Sheffer function (1913)	$\bar{T}_8$
T_{11}	1	0	1	1	$a + \bar{b} = \overline{\bar{a}.b}$, $b \rightarrow a$, implication	$\bar{T}_4$
T_{13}	1	1	0	1	$\bar{a} + b = \overline{a.\bar{b}}$, $a \rightarrow b$, implication	$\bar{T}_2$
T_{14}	1	1	1	0	$a + b$, OR, disjunction, INCLUSIVE–OR	$\bar{T}_1$
T_{15}	1	1	1	1	one function (closed, true)	$\bar{T}_0$

Table 1: All functions of 2 variables.

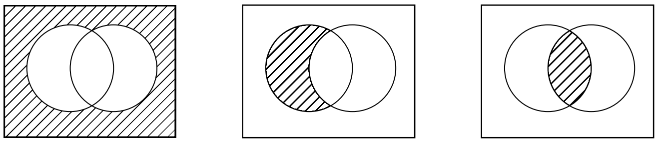
The most important functions are AND, NAND, OR, NOR and EXCLUSIVE–OR. Until now we have been using complete circuits. The symbols which are normally used are illustrated below:



We see that we can use a small circle for representing a negation. An invertor which generates the negation of the input signal is often symbolised by  or  .

The above basic functions, and many more, are available in *hardware.* These are integrated circuits in plastic or ceramic packages: chips. For example, one can buy a chip with 14 pins containing 4 NAND gates (14 pins because 4x3=12 for the NANDs plus two for the power supply: ground and e.g. +5V).

One can represent functions also in Venn diagrams as used in set theory:



$\bar{a} \cdot \bar{b}$ $\quad$ $a \cdot \bar{b}$ $\quad$ $a \cdot b$