3. Boole algebra

A Boole algebra is a set of elements $B = \{a, b, \ldots\}$ and two binary operators, i.e. operators defined between two elements. These operators are:

$\wedge$ meaning intersection, and
$\vee$ meaning union,

often, and also here, denoted by $\cdot$ (or .) and +. So $\wedge \equiv .$ and $\vee \equiv +$. These operators for the Boole algebra are defined as follows (so-called *postulates;* Huntington, 1904):

1. $\forall a, \forall b, \exists c \in B$ such that $a + b = c$
   $\forall a, \forall b, \exists d$ such that $a.b = d$

2. **Commutative law:** $\forall a, \forall b$
   2.1 $a + b = b + a$
   2.2 $a.b = b.a$

3. **Associative law:** $\forall a, \forall b, \forall c$
   3.1 $(a + b) + c = a + (b + c)$
   3.2 $(a.b).c = a.(b.c)$

4. **Distributive law:** $\forall a, \forall b, \forall c$
   4.1 $a + (b.c) = (a + b).(a + c)$
   4.2 $a.(b + c) = (a.b) + (a.c)$

5. $\forall a \in B, \exists! z$ such that $a + z = a$
   We will denote this element by the symbol 0, so $a + 0 = a$

6. $\forall a \in B, \exists! e$ such that $a.e = a$
   We will denote this element by the symbol 1, so $a.1 = a$

7. $\forall a \in B, \exists! \bar{a}$ such that $a.\bar{a} = 0$ and $a + \bar{a} = 1$
   This element is called the **inverse** of a, and also denoted $a\prime$, $\neg a$, or "not a"

Ad 5: z is unique.
Proof: Suppose $z_1 + a = a$ — Suppose $z_2 + a = a$
If $a = z_2$ then $z_1 + z_2 = z_2$ — If $a = z_1$ then $z_1 + z_2 = z_1$
Hence: $z_1 = z_1 + z_2 = z_2 \rightsquigarrow z_1 = z_2 = z$.

Ad 6: e is unique.
Proof: Suppose $e_1.a = a$ — Suppose $e_2.a = a$
If $a = e_2$ then $e_1.e_2 = e_2$ — If $a = e_1$ then $e_1.e_2 = e_1$
Hence: $e_1 = e_1.e_2 = e_2 \rightsquigarrow e_1 = e_2 = e$.

Ad 7: $\bar{a}$ is unique.
Proof: Suppose there are $\bar{a}_1$ and $\bar{a}_2$

$a + \bar{a}_1 = 1$	$a + \bar{a}_2 = 1$
$a.\bar{a}_1 = 0$	$a.\bar{a}_2 = 0$

Continue yourself with **6** *and* **4.2**!

Results of the postulates (corollaries)

R1:
$a.a = a.a + 0 = a.a + a.\bar{a} = a.(a + \bar{a})$ (see 4.2)
$= a.1 = a$. We say that a is *idempotent:* $\underline{\ a.a = a\ }$

R2:
$a + a = (a + a).1 = (a + a).(a + \bar{a}) = a + (a.\bar{a})$ (see 4.1)
$= a + 0 = a$: $\underline{\ a + a = a\ }$

R3:
$a + 1 = a + (a + \bar{a}) = (a + a) + \bar{a} = a + \bar{a} = 1$: $\underline{\ a + 1 = 1\ }$

R4:
$a.0 = a.(a.\bar{a}) = (a.a).\bar{a} = a.\bar{a} = 0$: $\underline{\ a.0 = 0\ }$

R5:
$a + a.b = a.1 + a.b = a.(1 + b) = a.1 = a$: $\underline{\ a + a.b = a\ }$

R6:
$a.(a + b) = a.a + a.b = a + a.b = a$ (use R5): $\underline{\ a.(a + b) = a\ }$
R5 and R6 are the so-called *absorption theorems.*

R7:
$a + \bar{a}.b = (a + \bar{a}).(a + b)$ (see 4.1)
$= 1.(a + b)$: $\underline{\ a + \bar{a}b = a + b\ }$

R8:
$a.(\bar{a} + b) = a.\bar{a} + a.b = 0 + a.b = a.b$: $\underline{\ a.(\bar{a} + b) = a.b\ }$

R9:
$a'' = a$
Proof: Substitute $(a')'$ for b in R7.
This gives $a + (a')' = a + a'.(a')' = a + 0 = a$
Now substitute $(a')'$ for b in R8.
This gives $a.(a')' = a.(a' + (a')') = a.1 = a$. Hence $\underline{\ (a')' = a\ }$

R10:
$(a + b)' = ?$
$(a + b).(\bar{a}.\bar{b}) = [a.(\bar{a}.\bar{b})] + [b.(\bar{a}.\bar{b})] = a.\bar{a}.\bar{b} + b.\bar{b}.\bar{a} = 0$
$(a+b)+\bar{a}.\bar{b} = [(a+b)+\bar{a}].[(a+b)+\bar{b}] = [(a+\bar{a})+b].[a+(b+\bar{b})] = (1+b).(a+1) = 1.1 = 1$
So $\bar{a}.\bar{b}$ is apparently the inverse of $a + b$: $\underline{\ \bar{a}.\bar{b} = (a + b)'\ }$
In general we have **R10a:**

$$(a + b + c + \cdots)' = \bar{a}.\bar{b}.\bar{c}.\cdots$$

R11:
$(a.b)\prime = ?$
$(a.b).(\bar{a}+\bar{b}) = a\bar{a}b + ab\bar{b} = 0$
$(a.b) + \bar{a} + \bar{b} = (\bar{a}+\bar{b}+a).(\bar{a}+\bar{b}+b)$ (see 4.1)
$= 1.1 = 1.$
So $\bar{a}+\bar{b}$ is apparently the inverse of $a.b$: $\underline{\bar{a}+\bar{b} = (a.b)\prime}$
In general we have **R11a:**

$$(a.b.\cdots)\prime = \bar{a}+\bar{b}+\bar{c}+\cdots$$

R12:
$(\sum_{i=1}^{n} a_i b_i)\prime = ?$
$(\sum_i a_i b_i)\prime = (a_1b_1 + a_2b_2 + a_3b_3 + \cdots)\prime$
Take $a_1b_1 = x$, $a_2b_2 = y$, $a_3b_3 = z$, etc.
$(x+y+z+\cdots)\prime = \bar{x}.\bar{y}.\bar{z}.\cdots$ (see R10a)
$= (a_1b_1)\prime.(a_2b_2)\prime.(a_3b_3)\prime.\cdots$
$= (\bar{a}_1+\bar{b}_1).(\bar{a}_2+\bar{b}_2).(\bar{a}_3+\bar{b}_3).\cdots.$ (see R11a)
Hence

$$(\sum_i a_i b_i)\prime = \prod_i (\bar{a}_i + \bar{b}_i)$$

The inverse of a sum of products is the product of the sums of the inverse elements!

Similarly (prove yourself!):
R13:

$$[\prod_i (a_i + b_i)]\prime = \sum_i \bar{a}_i \bar{b}_i$$

The inverse of a product of sums is the sum of the products of the inverse elements. R12 and R13 are the theorems of *de Morgan* (1847).

R14:
- $ab + \bar{a}c + bc =$
$ab1 + \bar{a}c1 + bc1 =$
$ab1 + \bar{a}c1 + bc(a+\bar{a}) =$
$ab1 + \bar{a}c1 + bca + bc\bar{a} =$
$ab(1+c) + \bar{a}c(1+b) =$
$ab + \bar{a}c$
The term bc has been *absorbed; bc is redundant.*

- $(a+b)(\bar{a}+c)(b+c) =$
$(a+b+0)(\bar{a}+c+0)(b+c+a\bar{a}) =$
$(a+b+0)(\bar{a}+c+0)(a+b+c)(\bar{a}+b+c) =$
$[(a+b)+0.c].[(\bar{a}+c)+0.b] =$
$(a+b)(\bar{a}+c)$
The term $(b+c)$ has been absorbed. Hence, combining the last two results we have

$$(a+b)(\bar{a}+c) = ac + \bar{a}b$$

Duality
From all theory we notice that all rules come in pairs. Examples:

$a + 0 = a$
$a.1 = a$

and

$a + (b.c) = (a + b).(a + c)$
$a.(b + c) = (a.b) + (a.c)$

Every theorem which can be proven for the Boole algebra has a *dual* form which also holds! The four conversion rules are:

$. \rightarrow +$
$+ \rightarrow .$
$1 \rightarrow 0$
$0 \rightarrow 1$

Finally, we will often work with the **binary set** $B = \{0, 1\}$.
$a.1 = a$ implies $1.1 = 1$ $(a = 1)$ and $0.1 = 0$ $(a = 0)$
$a + 0 = a$ implies $1 + 0 = 1$ $(a = 1)$ and $0 + 0 = 0$ $(a = 0)$